

Exact Rosenthal-type inequalities for $p = 3$, and related results

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Abstract: An exact Rosenthal-type inequality for the third absolute moments is given, as well as a number of related results. Such results are useful in applications to Berry–Esseen bounds.

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1. Introduction, summary, and discussion

Let X_1, \dots, X_n be independent random variables (r.v.'s), with the sum $S := X_1 + \dots + X_n$, such that for some real positive constant β and all i one has

$$\mathbb{E} X_i \leq 0, \quad \sum \mathbb{E} X_i^2 \leq 1, \quad \text{and} \quad \sum \mathbb{E}(X_i)_+^3 \leq \beta; \quad (1)$$

as usual, we let $x_+ := 0 \vee x$ and $x_+^p := (x_+)^p$ for all real x and all real $p > 0$.

Consider the following class of functions:

$$\begin{aligned} \mathcal{F}^3 &:= \{f \in \mathcal{C}^2: f \text{ and } f'' \text{ are nondecreasing and convex}\} \\ &= \{f \in \mathcal{C}^2: f, f', f'', f''' \text{ are nondecreasing}\}, \end{aligned} \quad (2)$$

where \mathcal{C}^2 denotes the class of all twice continuously differentiable functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and f''' denotes the right derivative of the convex function f'' . For example, functions $x \mapsto a + bx + c(x - t)_+^\alpha$ and $x \mapsto a + bx + ce^{\lambda x}$ belong to \mathcal{F}^3 for all $a \in \mathbb{R}$, $b \geq 0$, $c \geq 0$, $t \in \mathbb{R}$, $\alpha \geq 3$, and $\lambda \geq 0$.

Remark. If a r.v. X has a finite expectation and a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is in \mathcal{F}^3 or, more generally, is any convex function, then, by Jensen's inequality, $\mathbb{E} f(X)$ always exists in $(-\infty, \infty]$.

The main result of this note is

Theorem 1. *For any function $f \in \mathcal{F}^3$*

$$\mathbb{E} f(S) \leq \mathbb{E} f(Z) + \frac{f'''(\infty-)}{3!} \beta, \quad (3)$$

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where Z is a standard normal r.v. Moreover, for each function $f \in \mathcal{F}^3$ the upper bound in (3) is exact, in the sense that it is equal to the supremum of $\mathbf{E} f(S)$ over all independent X_i 's satisfying conditions (1).

Of course, in the case when $f'''(\infty-) = \infty$, the inequality (3) is trivial. Theorem 1 is based on the main result of [10].

It follows immediately from Theorem 1 that for all real x

$$\mathbf{E}(S - x)_+^3 \leq \mathbf{E}(Z - x)_+^3 + \beta. \quad (4)$$

If it is additionally assumed that $\mathbf{E} X_i = 0$ for all i , then (4) in turn yields

$$\mathbf{E} |S - x|^3 \leq \mathbf{E} |Z - x|^3 + \sum \mathbf{E} |X_i|^3; \quad (5)$$

moreover, one can similarly show that the upper bound in (5) is exact, for each real x ; the special case $x = 0$ of (5) is also a special case of Rosenthal's inequality [16]:

$$\mathbf{E} |S|^p \leq c_p \left(1 + \sum \mathbf{E} |X_i|^p\right), \quad (6)$$

for all $p \geq 2$, where c_p is a positive constant depending only on p (inequality (6) too needs the assumption that the X_i 's be zero-mean). In the case when $x = 0$ and the X_i 's are symmetric, inequality (5) was obtained by Ibragimov and Sharakhmetov [4], who at that considered arbitrary real $p > 2$. Besides, inequality (5) follows from Tyurin's result [17, Theorem 2], which also implies (4) but with $\sum \mathbf{E} |X_i|^3$ in place of β . More on Rosenthal-type inequalities and related results can be found, among other papers, in [1–3, 5–8, 11, 15, 18].

Theorem 1 admits

Corollary 2. *For any $p \in (0, 3)$ and any real $a > 0$*

$$\mathbf{E} S_+^p \leq \frac{p^p(3-p)^{3-p}}{3^3} \frac{\mathbf{E}(Z + a)_+^3 + \beta}{a^{3-p}};$$

in particular, taking here $(p, a) = (1, \frac{1746}{1000})$ and $(p, a) = (2, \frac{639}{1000})$, one obtains, respectively, the inequalities

$$\mathbf{E} S_+ \leq 0.514 + 0.0486\beta \quad \text{and} \quad \mathbf{E} S_+^2 \leq 0.555 + 0.232\beta. \quad (7)$$

One may compare the latter two bounds with the “naive” ones, obtained using the inequalities $(\mathbf{E} S_+)^2 \leq \mathbf{E} S_+^2 \leq \mathbf{E} S^2 \leq 1$; here one may note that β will rather typically be small. One can similarly bound $\mathbf{E}(S - x)_+^p$ for any real x and any $p \in (0, 3)$. The first inequality in (7) can in fact be improved:

$$\mathbf{E} S_+ \leq \frac{1}{2}, \quad (8)$$

which follows because $4u_+ \leq u^2 + 2u + 1$ for all real u ; the bound $\frac{1}{2}$ on $\mathbf{E} S_+$ in (8) is obviously attained when $\mathbf{P}(S = \pm 1) = \frac{1}{2}$.

The case $p = 3$ of Rosenthal-type inequalities, including the results stated above, is especially important in applications to Berry–Esseen bounds; see e.g. [14], Remark 3.4 in [12], and the “quick proofs” of Nagaev's nonuniform Berry–Esseen bound in [12, 13].

2. Proofs

Proof of Theorem 1. Take indeed any $f \in \mathcal{F}^3$. Next, take any real $y > \beta$ and introduce the r.v.'s

$$X_{i,y} := X_i \wedge y \quad \text{and} \quad S_y := \sum_i X_{i,y}.$$

Then the conditions (1) hold for the $X_{i,y}$'s in place of X_i . Also, $X_{i,y} \leq y$ for all i . So, by the main result of [10],

$$\mathbb{E} f(S_y) \leq \mathbb{E} f(\sqrt{1 - \beta/y} Z + y \tilde{\Pi}_{\beta/y^3}) \quad (9)$$

$$= \sum_{j=0}^{\infty} \mathbb{E} f(\sqrt{1 - \beta/y} Z + yj - \beta/y^2) \frac{(\beta/y^3)^j}{j!} e^{-\beta/y^3}, \quad (10)$$

where $\tilde{\Pi}_\theta := \Pi_\theta - \mathbb{E} \Pi_\theta = \Pi_\theta - \theta$ and Π_θ is any r.v. which is independent of Z and has the Poisson distribution with parameter θ , for any real $\theta > 0$. Moreover, by [10, Proposition 2.3], for any given triple $(f, \beta, y) \in \mathcal{F}^3 \times (0, \infty) \times (0, \infty)$ with $y > \beta$ the bound in (9) is exact, in the sense that it is equal to the supremum of $\mathbb{E} f(S_y)$ over all independent X_i 's satisfying conditions (1).

Now let

$$y \rightarrow \infty.$$

Then, by the monotone convergence theorem,

$$\mathbb{E} f(S_y) \rightarrow \mathbb{E} f(S). \quad (11)$$

As was mentioned earlier, in the case when $f'''(\infty-) = \infty$ the inequality (3) is trivial. Consider now the case when $f'''(\infty-) < \infty$. Then, by a l'Hospital-type rule, $f(x)/x^3 \rightarrow f'''(\infty-)/3!$ as $x \rightarrow \infty$, which also leads to $|f(x)| = O(1 + |x|^3)$ over all real x (for negative real x , one even has $|f(x)| = O(1 + |x|)$, since f is nondecreasing and convex; cf. e.g. [9, Lemma 7]). Therefore, by the dominated convergence theorem,

$$\begin{aligned} \mathbb{E} f(\sqrt{1 - \beta/y} Z + yj - \beta/y^2) &\longrightarrow \mathbb{E} f(Z) & \text{if } j = 0 \\ \frac{\mathbb{E} f(\sqrt{1 - \beta/y} Z + yj - \beta/y^2)}{y^3} &\longrightarrow f'''(\infty-) \frac{j^3}{3!} & \text{if } j > 0, \end{aligned}$$

and so, again by the dominated convergence theorem (say), the sum in (10) converges to $\mathbb{E} f(Z) + \frac{f'''(\infty-)}{3!} \beta$. In view of (9)–(11), this proves the inequality (3); the exactness of the bound in (3) follows from that of the bound in (9)–(10). \square

Proof of Corollary 2. This follows from (4), since $\sup_{u \geq 0} \frac{u^p}{(u+a)^3} = \frac{p^p(3-p)^{3-p}}{3^3 a^{3-p}}$ for any $p \in (0, 3)$ and any real $a > 0$. \square

References

- [1] S. Boucheron, O. Bousquet, G. Lugosi, and P. Massart. Moment inequalities for functions of independent random variables. *Ann. Probab.*, 33(2):514–560, 2005.
- [2] D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42, 1973.
- [3] E. Giné, R. Latała, and J. Zinn. Exponential and moment inequalities for U -statistics. In *High dimensional probability, II (Seattle, WA, 1999)*, volume 47 of *Progr. Probab.*, pages 13–38. Birkhäuser Boston, Boston, MA, 2000.
- [4] R. Ibragimov and S. Sharakhmetov. On an exact constant for the Rosenthal inequality. *Teor. Veroyatnost. i Primenen.*, 42(2):341–350, 1997.
- [5] R. Ibragimov and S. Sharakhmetov. On extremal problems and best constants in moment inequalities. *Sankhyā Ser. A*, 64(1):42–56, 2002.
- [6] R. Latała. Estimation of moments of sums of independent real random variables. *Ann. Probab.*, 25(3):1502–1513, 1997.
- [7] S. V. Nagaev and I. F. Pinelis. Some inequalities for the distributions of sums of independent random variables. *Teor. Veroyatnost. i Primenen.*, 22(2):254–263, 1977. MR0443034.
- [8] I. Pinelis. Optimum bounds on moments of sums of independent random vectors. *Siberian Adv. Math.*, 5(3):141–150, 1995.
- [9] I. Pinelis. Exact inequalities for sums of asymmetric random variables, with applications. *Probab. Theory Related Fields*, 139(3-4):605–635, 2007.
- [10] I. Pinelis. On the Bennett-Hoeffding inequality, a shorter version to appear in *Annales de l’Institut Henri Poincaré*. <http://arxiv.org/abs/0902.4058>, 2012.
- [11] I. Pinelis. Rosenthal-type inequalities for martingales in 2-smooth Banach spaces. <http://arxiv.org/abs/1212.1912>, 2012.
- [12] I. Pinelis. More on the nonuniform Berry–Esseen bound. <http://arxiv.org/abs/1302.0516>, 2013.
- [13] I. Pinelis. On the nonuniform Berry–Esseen bound. <http://arxiv.org/abs/1301.2828>, 2013.
- [14] I. Pinelis and R. Molzon. Berry–Esseen bounds for general nonlinear statistics, with applications to Pearson’s and non-central Student’s and Hotelling’s (preprint), arXiv:0906.0177 [math.ST].
- [15] I. F. Pinelis and S. A. Utev. Estimates of moments of sums of independent random variables. *Theory Probab. Appl.*, 29(3):574–577, 1984.
- [16] H. P. Rosenthal. On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables. *Israel J. Math.*, 8:273–303, 1970.
- [17] I. S. Tyurin. Some optimal bounds in the central limit theorem using zero biasing. *Statist. Probab. Lett.*, 82(3):514–518, 2012.
- [18] S. A. Utev. Extremal problems in moment inequalities. In *Limit theorems of probability theory*, volume 5 of *Trudy Inst. Mat.*, pages 56–75, 175. “Nauka” Sibirsk. Otdel., Novosibirsk, 1985.